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THE TRANSVERSE SPACE-CHARGE FORCE IN TRI-GAUSSIAN DISTRIBUTION

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1 INTRODUCTION

In tracking, the transverse space-charge force can be represented by changes in the horizontal and vertical divergences, $\Delta x'$ and $\Delta y'$ at many locations around the accelerator ring. In this note, we are going to list some formulas for $\Delta x'$ and $\Delta y'$ arising from space-charge kicks when the beam is tri-Gaussian distributed. We will discuss separately a flat beam and a round beam.

We are not interested in the situation when the emittance growth arising from space charge becomes too large and the shape of the beam becomes weird. For this reason, we can assume the bunch still retains its tri-Gaussian distribution, with its rms sizes σ_x, σ_y, and σ_z increasing by certain factors. Thus after each turn, σ_x, σ_y, and σ_z can be re-calculated. The electric potential for a particle of charge e at location x, y, z is therefore given by the formula, [1]

$$U_{sc}(x,y,z) = \frac{eN}{4\pi^{3/2}\epsilon_0}\int_0^\infty dt\,\frac{\exp\left[-\frac{x^2}{2\sigma_x^2+t}-\frac{y^2}{2\sigma_y^2+t}-\frac{z^2}{2\sigma_z^2+t}\right]-1}{\sqrt{(2\sigma_x^2+t)(2\sigma_y^2+t)(2\sigma_z^2+t)}}, \tag{1}$$

where N is the number of particles in the bunch. The force acting on the particle is just $-e\vec{\nabla}U$. Written in this way, the potential vanishes at the center of the bunch. To include

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the magnetic part of the force, we just need to add a γ^2 in the denominator. There is no closed form for this integral.

2 Linear Approximation

When incorporating this into a code, it will be nice if there is an option of including only the linear part of the space-charge force. Take the x-component of the force. After performing the derivative with respect to x, the linear part can be obtained by setting $x = y = 0$ in the integrand. Thus

$$F_{x\,\mathrm{linear}} = -e\frac{\partial U_{sc}}{\partial x}\bigg|_{\mathrm{linear}} = \frac{e^2 N x}{2\pi^{3/2}\epsilon_0\gamma^2}\int_0^\infty dt\,\frac{\exp\left[-\frac{z^2}{2\sigma_z^2+t}\right]}{(2\sigma_x^2+t)^{3/2}(2\sigma_y^2+t)^{1/2}(2\sigma_z^2+t)^{1/2}}. \tag{2}$$

We can be sure that the algebraic sign on the right side is correct, because the space-charge force is repulsive and therefore positive in the positive x-direction. So far as we know, this integral cannot be performed in the closed form. However, since σ_y and σ_x are both very much smaller than σ_z in the damping ring, an approximation can be made. Notice that the integration variable t varies mostly between 0 and $2\sigma_x^2$. We can therefore make the replacement

$$2\sigma_z^2 + t \longrightarrow 2\sigma_z^2. \tag{3}$$

The above reduces to

$$F_{x\,\mathrm{linear}} = \frac{e^2 N x}{2\pi\epsilon_0\gamma^2}\frac{e^{-z^2/(2\sigma_z^2)}}{\sqrt{2\pi}\sigma_z}\int_0^\infty dt\,\frac{1}{(2\sigma_x^2+t)^{3/2}(2\sigma_y^2+t)^{1/2}}. \tag{4}$$

This integral can now be performed in the closed form to give the familiar result

$$F_{x\,\mathrm{linear}} = -e\frac{\partial U_{sc}}{\partial x}\bigg|_{\mathrm{linear}} = \frac{e^2 x}{2\pi\epsilon_0\gamma^2}\frac{1}{\sigma_x(\sigma_x+\sigma_y)}\frac{N e^{-z^2/(2\sigma_z^2)}}{\sqrt{2\pi}\sigma_z}. \tag{5}$$

In the same way, the linear part of the vertical space charge on the particle is

$$F_{y\,\mathrm{linear}} = -e\frac{\partial U_{sc}}{\partial y}\bigg|_{\mathrm{linear}} = \frac{e^2 y}{2\pi\epsilon_0\gamma^2}\frac{1}{\sigma_y(\sigma_x+\sigma_y)}\frac{N e^{-z^2/(2\sigma_z^2)}}{\sqrt{2\pi}\sigma_z}. \tag{6}$$

The equations of motion in the transverse planes are

$$\begin{aligned} x'' + K_x x &= \frac{F_x}{\gamma m v^2} = -\frac{e}{\gamma m v^2}\frac{\partial U_{sc}}{\partial x}, \\ y'' + K_y y &= \frac{F_y}{\gamma m v^2} = -\frac{e}{\gamma m v^2}\frac{\partial U_{sc}}{\partial y}, \end{aligned} \tag{7}$$

where $K_x x$ and $K_y y$ are restoring forces from the magnetic elements, while $F_{x,y}$ are the horizontal and vertical forces arising from space-charge effects. In above, m is the electron mass and $v = \beta c$ is the nominal velocity of the beam particles. Integrating over a length L of the orbit where the beam radii do not change much, we obtain the changes in x' and y' coming from space-charge effects only,

$$\begin{aligned}\Delta x' &= -\frac{eL}{\gamma m v^2}\frac{\partial U_{sc}}{\partial x},\\ \Delta y' &= -\frac{eL}{\gamma m v^2}\frac{\partial U_{sc}}{\partial y}.\end{aligned} \tag{8}$$

Thus, the linearized space-charge force leads to

$$\begin{aligned}\Delta x' &= \frac{2Nr_0 L e^{-z^2/(2\sigma_z^2)}}{\gamma^3\beta^2\sqrt{2\pi}\sigma_z}\frac{x}{\sigma_x(\sigma_x+\sigma_y)},\\ \Delta y' &= \frac{2Nr_0 L e^{-z^2/(2\sigma_z^2)}}{\gamma^3\beta^2\sqrt{2\pi}\sigma_z}\frac{y}{\sigma_y(\sigma_x+\sigma_y)},\end{aligned} \tag{9}$$

where $r_0 = e^2/(4\pi\epsilon_0 mc^2)$ represents the classical radius of the beam particle. They can also be rewritten as

$$\begin{aligned}\Delta x' &= \frac{K_{sc} L e^{-z^2/(2\sigma_z^2)}}{\sqrt{2\pi}\sigma_z}\frac{x}{\sigma_x(\sigma_x+\sigma_y)},\\ \Delta y' &= \frac{K_{sc} L e^{-z^2/(2\sigma_z^2)}}{\sqrt{2\pi}\sigma_z}\frac{y}{\sigma_y(\sigma_x+\sigma_y)}.\end{aligned} \tag{10}$$

where

$$K_{sc} = \frac{2Nr_0}{\gamma^3\beta^2}, \tag{11}$$

is called the space-charge perveance of the beam.

3 Application to Electron Bunches

An electron bunch usually has its length very much larger than the transverse radii. Thus the replacement in Eq. (3) can be made, and the space-charge potential becomes (including electric and magnetic contributions)

$$U_{sc}(x,y,z) = \frac{eN}{4\pi^{3/2}\epsilon_0\gamma^2}\frac{e^{-z^2/(2\sigma_z^2)}}{\sqrt{2}\sigma_z}\int_0^\infty dt\,\frac{\exp\left[-\frac{x^2}{2\sigma_x^2+t}-\frac{y^2}{2\sigma_y^2+t}\right]-1}{\sqrt{(2\sigma_x^2+t)(2\sigma_y^2+t)}}. \tag{12}$$

This integral can be performed in the closed form in terms of the complex error function. [2] As will be shown below, however, the analytic expression is useful only for electron bunches, where the horizontal beam radius is very much larger than the vertical ($\sigma_x \gg \sigma_y$).

Introduce the following new variables:

$$s^2 = \frac{2\sigma_y^2 + t}{2\sigma_x^2 + t}, \quad a = \frac{x}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}}, \quad b = \frac{y}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}}, \quad \text{and} \quad r = \frac{\sigma_y}{\sigma_x}. \tag{13}$$

It is then easy to obtain

$$U_{sc}(x,y,z) = \frac{e^2 N e^{-z^2/(2\sigma_z^2)}}{(2\pi)^{3/2}\epsilon_0 \gamma^2 \sigma_z} \int_r^1 \frac{ds}{1-s^2} \left[e^{-a^2(1-s^2) - b^2\left(\frac{1}{s^2}-1\right)} - 1 \right]. \tag{14}$$

We therefore have

$$\begin{aligned} \Delta x' &= \frac{K_{sc} L\, e^{-z^2/(2\sigma_z^2)}}{\sqrt{2\pi}\sigma_z} \frac{x}{\sigma_x^2 - \sigma_y^2} \int_r^1 ds\, e^{-a^2(1-s^2) - b^2\left(\frac{1}{s^2}-1\right)}. \\ \Delta y' &= \frac{K_{sc} L\, e^{-z^2/(2\sigma_z^2)}}{\sqrt{2\pi}\sigma_z} \frac{y}{\sigma_x^2 - \sigma_y^2} \int_r^1 \frac{ds}{s^2}\, e^{-a^2(1-s^2) - b^2\left(\frac{1}{s^2}-1\right)}. \end{aligned} \tag{15}$$

We understand that the signs before the right sides of these two equations are correct, because for positive x and/or y, the horizontal and/or vertical divergence should increase.

The changes in the horizontal and vertical divergences can be combined as a complex variable

$$\Delta x' - i\Delta y' = \frac{K_{sc} L\, e^{-z^2/(2\sigma_z^2)}}{\sqrt{2\pi}\sigma_z} \frac{2}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}} \int_r^1 ds \left(a - i\frac{b}{s^2} \right) e^{-(as+ib/s)^2 - (a+ib)^2}. \tag{16}$$

A new variable of integration

$$\zeta = as + i\frac{b}{s} \tag{17}$$

is now introduced, which simplifies the above to

$$\Delta x' - i\Delta y' = \frac{K_{sc} L\, e^{-z^2/(2\sigma_z^2)}}{\sqrt{\pi}\sigma_z} \frac{e^{-(a+jb)^2}}{\sqrt{\sigma_x^2 - \sigma_y^2}} \int_{ar+\frac{ib}{r}}^{a+ib} e^{\zeta^2} d\zeta. \tag{18}$$

From the definition of the complex error function

$$w(z) = e^{-z^2} \left[1 + \frac{2i}{\sqrt{\pi}} \int_0^z e^{\zeta^2} d\zeta \right], \tag{19}$$

we arrive at

$$\Delta x' - i\Delta y' = -i\frac{K_{sc}L\, e^{-z^2/(2\sigma_z^2)}}{2\sigma_z\sqrt{\sigma_x^2-\sigma_y^2}}\left[w(a+ib) - e^{-a^2(1-r^2)-b^2\left(\frac{1}{r^2}-1\right)}w\left(ar+i\frac{b}{r}\right)\right], \tag{20}$$

or

$$\begin{aligned}
\Delta x' &= \frac{K_{sc}L\, e^{-z^2/(2\sigma_z^2)}}{2\sigma_z\sqrt{\sigma_x^2-\sigma_y^2}}\,\mathcal{Im}\left[w\left(\frac{x+iy}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right) - e^{-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}}\,w\left(\frac{x\frac{\sigma_x}{\sigma_y}+iy\frac{\sigma_y}{\sigma_x}}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right)\right],\\
\Delta y' &= \frac{K_{sc}L\, e^{-z^2/(2\sigma_z^2)}}{2\sigma_z\sqrt{\sigma_x^2-\sigma_y^2}}\,\mathcal{Re}\left[w\left(\frac{x+iy}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right) - e^{-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}}\,w\left(\frac{x\frac{\sigma_x}{\sigma_y}+iy\frac{\sigma_y}{\sigma_x}}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right)\right].
\end{aligned} \tag{21}$$

These expressions appear to diverge when $\sigma_x = \sigma_y$ because of the factors outside the squared-brackets. In fact, this is not true, because the expressions inside the square brackets will provide zeros at $\sigma_x = \sigma_y$ to cancel the poles outside. This is obvious, because the original expression for the space-charge potential in Eq. (1) is well-behaved at $\sigma_x = \sigma_y$.

4 ALMOST ROUND BEAM

It is obvious that the expressions in terms of the complex error function in Eq. (21) cannot be applied when the beam is almost round because of the singularities at $\sigma_x = \sigma_y$ outside and within the square brackets. Instead, let us start our discussion from Eq. (15), which can be rewritten as

$$\begin{aligned}
\Delta x' &= \frac{K_{sc}L\, e^{-z^2/(2\sigma_z^2)}}{\sqrt{2\pi}\sigma_z}\frac{x}{\sigma_x^2}f_x(x^2,y^2),\\
\Delta y' &= \frac{K_{sc}L\, e^{-z^2/(2\sigma_z^2)}}{\sqrt{2\pi}\sigma_z}\frac{y}{\sigma_y^2}f_y(x^2,y^2),
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
f_x(x^2,y^2) &= \frac{1}{1-r^2}\int_r^1 ds\, e^{-a^2(1-s^2)-b^2\left(\frac{1}{s^2}-1\right)},\\
f_y(x^2,y^2) &= \frac{r^2}{1-r^2}\int_r^1 \frac{ds}{s^2}\, e^{-a^2(1-s^2)-b^2\left(\frac{1}{s^2}-1\right)}.
\end{aligned} \tag{23}$$

where $r = \sigma_y/\sigma_x$. We may perform power expansion in x^2 and y^2 to obtain

$$f_x(x^2, y^2) = \frac{1}{1+r}\left[1 - \left(\frac{x^2}{2\sigma_x^2}\right)\frac{2+r}{3(1+r)} - \left(\frac{y^2}{2\sigma_y^2}\right)\frac{r}{1+r} + \left(\frac{x^2}{2\sigma_x^2}\right)^2 \frac{8+9r+3r^2}{30(1+r)^2} + \right.$$
$$\left. + \left(\frac{y^2}{2\sigma_y^2}\right)^2 \frac{r(1+3r)}{6(1+r)^2} + \left(\frac{x^2}{2\sigma_x^2}\right)\left(\frac{y^2}{2\sigma_y^2}\right)\frac{r(3+r)}{3(1+r)^2} + \cdots \right],$$
$$f_y(x^2, y^2) = \frac{r}{1+r}\left[1 - \left(\frac{x^2}{2\sigma_x^2}\right)\frac{1}{1+r} - \left(\frac{y^2}{2\sigma_y^2}\right)\frac{1+2r}{3(1+r)} + \left(\frac{x^2}{2\sigma_x^2}\right)^2 \frac{3+r}{6(1+r)^2} + \right.$$
$$\left. + \left(\frac{y^2}{2\sigma_y^2}\right)^2 \frac{3+9r+8r^2}{30(1+r)^2} + \left(\frac{x^2}{2\sigma_x^2}\right)\left(\frac{y^2}{2\sigma_y^2}\right)\frac{1+3r}{3(1+r)^2} + \cdots \right]. \tag{24}$$

It is evident that $f_y(x^2, y^2)$ can be obtained from $f_x(x^2, y^2)$ by suitable replacements of r by $1/r$. Unfortunately, these formulas may not be very useful, because many more terms in the power series will be necessary when a particle is near the edge of the beam, for example, at $x = \sqrt{6}\sigma_x$ and/or $y = \sqrt{6}\sigma_y$.

If the beam is nearly round, we can write $f_{x,y}(x^2, y^2)$ as functions of r and expand around $r = 1$. The result is

$$f_{x,y}(x^2, y^2) = \frac{\sigma_x^2}{x^2+y^2}\left[1 - e^{-\frac{x^2+y^2}{2\sigma_x^2}}\right] + \mathrm{Rem}\big(f_{x,y}\big). \tag{25}$$

The first term is just $f_{x,y}$ at $r = 1$, which is the expression easily derivable from Gauss's law when the beam is exactly round. The first derivative vanishes ($df_{x,y}/dr = 0$ at $r = 1$). Unfortunately, the second derivative is divergent at $r = 1$. As a result, Eq. (25) can be considered as a Taylor expansion with $\mathrm{Rem}\big(f_{x,y}\big)$ of order $\big(1 - \sigma_y^2/\sigma_x^2\big)^2$, representing the remainder[†] after the first derivative.

References

[1] A derivation is available at K. Takayama, Lett. Al Nuovo Cimento **34**, 190 (1982).

[†]In one representation,

$$\mathrm{Rem}\big(f_x\big) = \left(1 - \frac{\sigma_y^2}{\sigma_x^2}\right)^2 \frac{(1 - 2y^2/\sigma_x^2)e^{-\frac{x^2+y^2}{2\sigma_x^2}}}{8\epsilon}, \tag{26}$$

where ϵ is some number satisfying $0 < \epsilon < 1 - \sigma_y^2/\sigma_x^2$.

[2] M. Bassetti and G.A. Erskine, *Closed Expression for the Electrical Field of a Two-Dimensional Gaussian Charge*, CERN Report CERN-ISR-TH/80-06, 1980.